

## A short table of generating functions and related formulas

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A table of sums useful for generating function applications (discrete Laplace transforms or  $z$ -transforms). Related definitions and formulas (including Lagrange's expansion), and reference to formulas in Abramowitz and Stegun *Handbook of Mathematical Functions* are given.

Bold-face equation numbers refer to formulas in Abramowitz and Stegun. Many of these can be found or verified using Mathematica.

### Notation and Definitions of Functions

In general,  $z$ ,  $r$ , and  $s$  are complex numbers (although some formulas may be valid for real values only),  $x$  a real number,  $n$  and  $m$  are integers,  $\mathcal{R}z$  means the real part of  $z$ .

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = z(z+1)\dots(z+n-1), \quad (z)_0 = 0 \quad \mathbf{6.1.22}$$

$$\binom{r}{n} = \frac{r(r-1)(r-2)\dots(r-n+1)}{n!}, \quad \binom{m}{n} = \frac{m!}{n!(m-n)!} \quad (m \geq n) \quad \mathbf{24.1.1}$$

### Sums and Functions

Binomial formula (in general, valid for  $|z| < 1$ ):

$$\sum_{n=0}^{\infty} \binom{r}{n} z^n = (1+z)^r \quad \mathbf{3.6.8}$$

$$\sum_{n=0}^{\infty} \binom{n+s-1}{n} z^n = \sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{\Gamma(s)} \frac{z^n}{n!} = \sum_{n=0}^{\infty} (s)_n \frac{z^n}{n!} = (1-z)^{-s} \quad \mathbf{3.6.9}$$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \mathbf{3.6.10}$$

$$\sum_{n=0}^{\infty} n z^n = z \frac{d}{dz} \sum_{n=0}^{\infty} z^n = z \frac{d}{dz} \frac{1}{1-z} = \frac{z}{(1-z)^2}$$

$$\sum_{n=0}^{\infty} n^2 z^n = \left( z \frac{d}{dz} \right)^2 \frac{1}{1-z} = z \frac{d}{dz} \frac{z}{(1-z)^2} = \frac{z+z^2}{(1-z)^3}$$

$$\sum_{n=0}^{\infty} n^3 z^n = \frac{z+4z^2+z^3}{(1-z)^4}$$

$$\sum_{n=0}^{\infty} n^4 z^n = \frac{z + 11z^2 + 11z^3 + z^4}{(1-z)^5}$$

$$\sum_{n=0}^{\infty} n(n-1)z^n = 2 \sum_{n=0}^{\infty} \binom{n}{2} z^n = z^2 \left( \frac{d}{dz} \right)^2 \sum_{n=0}^{\infty} z^n = \frac{2z^2}{(1-z)^3}$$

$$\frac{1}{i!} \sum_{n=0}^{\infty} [n(n-1)(n-2) \dots (n-i+1)] z^n = \sum_{n=0}^{\infty} \binom{n}{i} z^n = \frac{z^i}{(1-z)^{i+1}}$$

$$\sum_{n=0}^{\infty} (n+1)z^n = \frac{1}{(1-z)^2}$$

$$\frac{1}{2} \sum_{n=0}^{\infty} (n+1)(n+2)z^n = \frac{1}{(1-z)^3}$$

$$\frac{1}{i!} \sum_{n=0}^{\infty} [(n+1)(n+2) \dots (n+i)] z^n = \sum_{n=0}^{\infty} \binom{n+i}{i} z^n = \frac{1}{(1-z)^{i+1}}$$

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{2^{2n}} = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)_n \frac{z^n}{n!} = (1-z)^{-1/2}$$

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{2^{2n}(n+1)} = \frac{2[1 - (1-z)^{1/2}]}{z}$$

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{2^{2n}(n+1)(n+2)} = \frac{4[(1-z)^{3/2} - 1 + 3z/2]}{3z^2}$$

log, inverse trigonometric functions:

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = \int_0^z \left( \sum_{n=1}^{\infty} t^{n-1} \right) dt = \int_0^z \frac{dt}{1-t} = -\ln(1-z) \quad \mathbf{4.1.24}$$

$$\sum_{n=1}^{\infty} \frac{t^n}{n^2} = \int_0^z \left( \sum_{n=1}^{\infty} \frac{t^{n-1}}{n} \right) dz = - \int_0^z \frac{\ln(1-t)}{t} dt = \text{Euler's dilogarithm} = g_2(z) \quad \mathbf{27.7.1}$$

$$\sum_{n=0}^{\infty} \frac{(\pm 1)^n z^{2n+1}}{2n+1} = \int_0^z \frac{dt}{1 \mp t^2} = \begin{cases} \arctanh z = (1/2) \ln[(1+z)/(1-z)] & (-) \\ \arctan z & (+) \end{cases} \quad \mathbf{4.6.22, 33} \quad \mathbf{4.4.42}$$

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{(\pm 1)^n z^{2n+1}}{2^{2n}(2n+1)} = \int_0^z \frac{dt}{(1 \mp t^2)^{1/2}} = \begin{cases} \arcsin z & (+) \\ \operatorname{arcsinh} z = \ln[z + (z^2 + 1)^{1/2}] & (-) \end{cases} \quad \begin{matrix} 4.4.40 \\ 4.6.31 \end{matrix}$$

$$\sum_{n=0}^{\infty} \frac{(\pm 1)^n 2^{2n} (n!)^2 z^{2n}}{(n+1)(2n+1)!} = \begin{cases} \left( \frac{\arcsin z}{z} \right)^2 & (+) \\ \left( \frac{\operatorname{arcsinh} z}{z} \right)^2 & (-) \end{cases} \quad \begin{matrix} 4.4.40 \\ 4.6.31 \end{matrix}$$

Exponential, trigonometric, hyperbolic functions:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z \quad \begin{matrix} 4.2.1 \end{matrix}$$

$$\sum_{n=0}^{\infty} \frac{(\pm 1)^n z^{2n+1}}{(2n+1)!} = \begin{cases} \sinh z = (e^z - e^{-z})/2 & (+) \\ \sin z = (e^{iz} - e^{-iz})/2i & (-) \end{cases} \quad \begin{matrix} 4.5.62 \\ 4.3.65 \end{matrix}$$

$$\sum_{n=0}^{\infty} \frac{(\pm 1)^n z^{2n}}{(2n)!} = \begin{cases} \cosh z = (e^z + e^{-z})/2 & (+) \\ \cos z = (e^{iz} + e^{-iz})/2 & (-) \end{cases} \quad \begin{matrix} 4.5.63 \\ 4.3.66 \end{matrix}$$

Exponential and Fresnel Integrals

$$\sum_{n=1}^{\infty} \frac{(\pm 1)^n z^n}{n n!} = \int_0^z \left( \sum_{n=1}^{\infty} \frac{(\pm 1)^n t^{n-1}}{n!} \right) dt = \int_0^z \frac{e^{\pm t} - 1}{t} dt = \begin{cases} \operatorname{Ei}(z) - \gamma - \ln z & (+) \\ -E_1(z) - \gamma - \ln z & (-) \end{cases} \quad \begin{matrix} 5.1.10 \\ 5.1.11 \end{matrix}$$

$$\sum_{n=0}^{\infty} \frac{(\pm 1)^n z^{2n+1}}{(2n+1)(2n+1)!} = \int_0^z \frac{(\operatorname{sinh}) t}{t} dt = \begin{cases} \operatorname{Shi}(z) & (+) \\ \operatorname{Si}(z) & (-) \end{cases} \quad \begin{matrix} 5.2.3, 17 \\ 5.2.1, 19 \end{matrix}$$

$$\sum_{n=1}^{\infty} \frac{(\pm 1)^n z^{2n}}{2n(2n)!} = \int_0^z \frac{(\operatorname{cosh}) t - 1}{t} dt = \begin{cases} \operatorname{Chi}(z) - \gamma - \ln z & (+) \\ \operatorname{Ci}(z) - \gamma - \ln z & (-) \end{cases} \quad \begin{matrix} 5.2.4, 18 \\ 5.2.2, 16 \end{matrix}$$

Incomplete gamma function and error function

$$e^{-z} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(a+n+1)} = \frac{1}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{(a+n)n!} = \gamma^*(a, z) \quad 6.5.29$$

$$e^{-z^2} \sum_{n=0}^{\infty} \frac{2^{2n} n! z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)n!} = \frac{\sqrt{\pi}}{2} \operatorname{erf} z = \frac{\sqrt{\pi}}{2} z \gamma^*(\frac{1}{2}, z^2) \quad 7.1.5$$

Bessel function

$$\sum_{n=0}^{\infty} \frac{(\pm 1)^n z^{2n}}{2^{2n} n! \Gamma(\nu + n + 1)} = \begin{cases} (2/z)^\nu I_\nu(z) & (+) \\ (2/z)^\nu J_\nu(z) & (-) \end{cases} \quad \mathbf{9.1.10}$$

$$\sum_{n=0}^{\infty} \frac{(\pm 1)^n z^{2n}}{2^{2n} n! n!} = \begin{cases} I_0(z) & (+) \\ J_0(z) & (-) \end{cases} \quad \mathbf{9.1.12}$$

Elliptic integrals

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{z^n}{2^{4n}} = \frac{2}{\pi} \int_0^{\pi/2} (1 - z \sin^2 \theta)^{1/2} d\theta = 2K(z)/\pi \quad \mathbf{17.3.1, 11}$$

$$\sum_{n=1}^{\infty} \binom{2n}{n}^2 \frac{z^n}{2^{4n}(2n-1)} = 1 - \frac{2}{\pi} \int_0^{\pi/2} (1 - z \sin^2 \theta)^{-1/2} d\theta = 1 - 2E(z)/\pi \quad \mathbf{17.3.2, 12}$$

Bernouilli functions and numbers [ $B_n(0) = B_n$ ,  $2^n E_n(\frac{1}{2}) = E_n$  – not the exponential-integral function].

$$\sum_{n=0}^{\infty} \frac{B_n(a) z^n}{n!} = \frac{ze^{az}}{e^z - 1} \quad |z| < 2\pi \quad \mathbf{23.1.1}$$

$$\sum_{n=0}^{\infty} \frac{E_n(a) z^n}{n!} = \frac{2e^{az}}{e^z + 1} \quad |z| < \pi \quad \mathbf{23.1.1}$$

Generalized zeta function (Bose functions) (see below)

$$\sum_{n=1}^{\infty} \frac{z^n}{n^s} = g_s(z)$$

Definitions of special functions in terms of integrals:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \quad (\Re z > 0) \quad \mathbf{6.1.1}$$

$$\Gamma(z+1) = z\Gamma(z), \Gamma(n) = n-1!, \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\gamma^*(a, x) = \frac{x^{-a}}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt = x^{-a} [1 - \Gamma(a, x)/\Gamma(a)] \quad \mathbf{6.5.4}$$

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t} dt \quad \mathbf{6.5.2}$$

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt \quad (|\arg z| < \pi) \quad \mathbf{5.1.1}$$

$$E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n} dt = z^{n-1} \Gamma(1-n, z) \quad (n = 0, 1, 2, \dots; \Re z > 0) \quad \mathbf{5.1.1}$$

$$\text{Ei}(x) = -\lim_{\epsilon \rightarrow 0} \left( \int_{-x}^{-\epsilon} \frac{e^{-t}}{t} dt + \int_\epsilon^\infty \frac{e^{-t}}{t} dt \right) \quad (x > 0) \quad \mathbf{5.1.2}$$

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad \mathbf{7.1.1}$$

$$J_\nu(z) = \frac{(z/2)^\nu}{\pi^{1/2} \Gamma(\nu + 1/2)} \int_0^\pi \cos(z \cos \theta) \sin^{2\nu} \theta d\theta \quad \mathbf{9.1.20}$$

$$I_\nu(z) = \frac{(z/2)^\nu}{\pi^{1/2} \Gamma(\nu + 1/2)} \int_0^\pi e^{\pm z \cos \theta} \sin^{2\nu} \theta d\theta \quad \mathbf{9.6.18}$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt \quad \Re s > 1 \quad \mathbf{23.2.7}$$

$$g_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} dt}{z^{-1} e^t - 1}$$

Generalized zeta functions, expansion for small  $\alpha = -\ln z \approx 1 - z$ :

$$g_s(z) = \begin{cases} \Gamma(1-s) \alpha^{n-1} + \sum_{k=0}^{\infty} \frac{\zeta(s-k)(-\alpha)^k}{k!} & s \neq 1, 2, 3, \dots \\ \frac{(-\alpha)^{s-1}}{(s-1)!} \left[ -\ln \alpha + \sum_{m=1}^{s-1} \frac{1}{m} \right] + \sum_{\substack{k=0 \\ k \neq s-1}}^{\infty} \frac{\zeta(s-k)(-\alpha)^k}{k!} & s = 1, 2, 3, \dots \end{cases}$$

Riemann zeta function  $[\zeta(0) = -1/2, \zeta(1) = \infty, \zeta(2) = \pi^2/6]$

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \quad \Re s > 1 \quad \mathbf{23.2.1}$$

Euler's gamma constant:

$$\gamma = \lim_{m \rightarrow \infty} \left[ \sum_{n=1}^m \frac{1}{n} - \ln m \right] = 0.57721 56649 01532 86060 \dots \quad \mathbf{6.1.3}$$

Taylor's expansion of  $f(z)$  about  $z_0 = 0$ :

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \left[ \frac{d^n}{dz^n} f(z) \right]_{z=0} \quad \mathbf{3.6.4}$$

### Lagrange's Expansion

Lagrange's expansion, where  $z = f(x)$ ,  $z_0 = f(x_0)$ ,  $f'(x_0) \neq 0$ :

$$x = f^{-1}(z) = x_0 + \sum_{n=1}^{\infty} \frac{(z - z_0)^n}{n!} \left[ \frac{d^{n-1}}{dx^{n-1}} \left\{ \frac{x - x_0}{f(x) - z_0} \right\}^n \right]_{x=x_0} \quad \mathbf{3.6.6}$$

For  $g(x)$  any infinitely differentiable function,

$$g(x) = g(0) + \sum_{n=1}^{\infty} \frac{(z - z_0)^n}{n!} \left[ \frac{d^{n-1}}{dx^{n-1}} \left( g'(x) \left\{ \frac{x - x_0}{f(x) - z_0} \right\}^n \right) \right]_{x=x_0} \quad \mathbf{3.6.7}$$

For special case of  $x_0 = 0$ ,  $z_0 = f(x_0) = 0$ :

$$x = \sum_{n=1}^{\infty} \frac{z^n}{n!} \left[ \frac{d^{n-1}}{dx^{n-1}} \left\{ \frac{x}{f(x)} \right\}^n \right]_{x=0} \quad \mathbf{3.6.6}$$

Examples: (Useful for polymerization, percolation on the Bethe lattice):

$$1 + r \sum_{n=1}^{\infty} \frac{(2n+r-1)! z^n}{(n+r)! n!} = 1 + r \sum_{n=1}^{\infty} (n+r-1)_{n-1} \frac{z^n}{n!} = r \sum_{n=0}^{\infty} \binom{2n+r}{n} \frac{z^n}{2n+r} = \left[ \frac{1 - (1-4z)^{1/2}}{2z} \right]^r$$

related:

$$\sum_{n=0}^{\infty} \binom{2n+r}{n} z^n = \frac{1}{\sqrt{1-4z}} \left[ \frac{1 - (1-4z)^{1/2}}{2z} \right]^r$$

Taking  $z = xe^{-x}$ :

$$\sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} (xe^{-x})^n = x$$

differentiating/integrating w.r.t.  $x$ :

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} (xe^{-x})^n = \frac{x}{1-x}$$

$$\sum_{n=1}^{\infty} \frac{n^{n+1}}{n!} (xe^{-x})^n = \frac{x}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} \frac{n^{n+2}}{n!} (xe^{-x})^n = \frac{x(1+2x)}{(1-x)^5}$$

$$\sum_{n=1}^{\infty} \frac{n^{n-2}}{n!} (xe^{-x})^n = x - \frac{x^2}{2}$$

$$\sum_{n=1}^{\infty} \frac{n^{n-3}}{n!} (xe^{-x})^n = x - \frac{3x^2}{4} + \frac{x^3}{6}$$

Combinatorial identities and relations:

$$\begin{aligned}
\binom{n - \frac{1}{2}}{n} &= \frac{\left(\frac{1}{2}\right)_n}{n!} = \frac{\Gamma(n + \frac{1}{2})}{\Gamma(\frac{1}{2})n!} = \frac{(2n-1)!!}{2^n n!} = \frac{(2n-1)!}{2^{2n-1} n!(n-1)!} = \frac{(2n)!}{2^{2n} (n!)^2} = \frac{1}{2^{2n}} \binom{2n}{n} \\
&= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (\sin x)^{2n} dx = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (\cos x)^{2n} dx \\
&\sim \frac{1}{(\pi n)^{1/2}} \exp \left( -\frac{1}{8n} - \frac{1}{192n^3} + \frac{1}{640n^5} \dots - \frac{B_{2m}(2^{2m}-1)}{2m(2m-1)2^{2m-1}n^{2m-1}} \dots \right) \\
&\sim \frac{1}{(\pi n)^{1/2}} \left[ 1 - \frac{1}{8n} + \frac{1}{128n^2} - \frac{1}{192n^3} \dots \right]
\end{aligned} \tag{6.1.49}$$

### Stirling's Approximation

$$\ln \Gamma(n) = \ln(n-1)! \sim n \ln n - n - \frac{1}{2} \ln n + \frac{1}{2} \ln(2\pi) + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} \dots \tag{6.1.41}$$

$$\ln n! \sim n \ln n - n + \frac{1}{2} \ln n + \frac{1}{2} \ln(2\pi) + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} \dots$$

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \dots\right)$$

### References

M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Government Printing Office, also Dover, and on the web at <http://www.math.sfu.ca/~cbm/aands/> [boldface equation numbers above refer to formulas in this book]. Updated web version (2010), see <http://dlmf.nist.gov/>.

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